Remarks on the Fact that the Uncertainty Principle Does Not Determine the Quantum State

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Abstract

We discuss the relation between density matrices and the uncertainty principle; this allows us to justify and explain a recent statement by Man'ko *et al.* We thereafter use Hardy's uncertainty principle to prove a new result for Wigner distributions dominated by a Gaussian and we relate this result to the coarse-graining of phase-space by "quantum blobs".

1 Introduction

Identifying the class of all phase-space functions who are Wigner distributions of some mixed quantum state is a formidable and unfinished task. The problem is actually the following: assume that a function $W(x,p)$ on phase space defines, via the Weyl correspondence, a self-adjoint operator $\widehat{\rho}$ with unit trace. Then $\widehat{\rho}$ is, a priori, an excellent candidate for being the density operator of some mixed state with Wigner distribution $W(x,p)$, provided that in addition this operator is *positive* (that is $\langle \widehat{\rho}\psi | \psi \rangle \geq 0$ for all square integrable ψ). And this is where the difficulty comes from: outside a few well-known cases (for instance when ρ is a Gaussian), it is notoriously difficult in general to check the positivity of $\widehat{\rho}$ by simply inspecting the function $W(x,p)$ (the condition $W(x,p) \geq 0$ is neither necessary nor sufficient, in strong opposition to the classical case). Although there are general (and difficult) mathematical theorems giving both necessary and sufficient conditions for positivity (the "KLM conditions", which we shortly review in Section 2), these results are not of great help in practice because they involve the simultaneous verification of the positivity of *infinitely* many square matrices of increasing dimension. A supplementary difficulty is actually lurking in the shadows: these conditions are sensitive to the value of Planck's constant when the latter is used as a variable parameter: a given operator $\widehat{\rho}$ might thus very well be positive for one value of $\hbar$ and negative for another (this somewhat unexpected but crucial property is best understood in terms of the Narcowich–Wigner spectrum [1, 2]). This feature is of course completely fatal when one wants to use semiclassical or WKB methods. In fact, in a recent

very interesting Letter [3] Man'ko *et al.* have shown that rescaling the position and momentum coordinates by a common factor can take a density matrix into a non-positive operator while preserving a class of sharp uncertainty relations (the Robertson–Schrödinger uncertainty principle, which we recall in Section 2). This allows these authors to conclude that "*...the uncertainty principle does not determine the quantum state*". Man'ko *et al.* are of course right; in fact Narcowich and O'Connell [4] had already shown in the mid 1980s that fulfilling the uncertainty relations is necessary, but *not sufficient*, to ensure the positivity of $\widehat{\rho}$ (this example is described in next Section).

The goal of this Letter is threefold. First (Section 2), we complement and explain from a somewhat different (and more critical) perspective the results of Man'ko *et al.* [3] (Man'ko *et al.* claim that the uncertainty relations are *necessary* to ensure positivity: they are again right, of course, but they do not prove this fundamental fact!). Secondly (Section 3) we propose a new criterion for deciding when a phase-space function which is dominated at infinity by a phase-space Gaussian is the Wigner distribution of a mixed (non necessarily Gaussian!) state. Our approach is based on Hardy's uncertainty principle [5] for a function and its Fourier transform. Hardy's theorem goes back to 1933: it is unfortunate that its usefulness in quantum mechanics has apparently not been noticed before! We conclude our discussion in Section 4) by linking our results to the notions of "quantum blob" and "admissible ellipsoid" introduced by the first author in [6, 7, 8], and which provides a canonically invariant notion of phase-space coarse-graining, which seems promising in various aspects of phase-space quantization.

Notation. We work in N degrees of freedom; the coordinates of position vector x are $x_1, ..., x_N$ and those of the momentum vector p are $p_1, ..., p_N$. Writing x and p as column vectors we set $z = \begin{pmatrix} x \\ p \end{pmatrix}$. We denote by $\sigma(z, z')$ the symplectic product: by definition $\sigma(z, z') = (z')^T Jz = p \cdot x' - p' \cdot x$ where where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard symplectic matrix. A $2N \times 2N$ real matrix S is symplectic if and only if $S^T JS = SJS^T = J$.

2 Canonical formulation of the uncertainty principle and positivity

Let $\widehat{\rho}$ be a self-adjoint trace-class operator on $L^2(\mathbb{R}^N)$; we have

$$\widehat{\rho}\psi(x) = \int K(x, x')\psi(x')d^N x' \tag{1}$$

where the kernel K satisfies $K(x, x') = \overline{K(x', x)}$ and is square integrable on $\mathbb{R}^N \times \mathbb{R}^N$. The Wigner distribution of $\widehat{\rho}$ is the real function

$$W(z) = \left(\tfrac{1}{2\pi\hbar}\right)^N \int e^{-\frac{i}{\hbar}p\cdot y} K(x + \tfrac{1}{2}y, x - \tfrac{1}{2}y) d^N y. \tag{2}$$

(see Littlejohn [9] for details; our choice of normalization is consistent with that adopted in Weyl calculus, but it differs from that in [3] even in the case $\hbar = 1$). We have

$$\mathrm{Tr}(\widehat{\rho}) = \int K(x,x)d^N x = \int W(z)d^{2N}z. \tag{3}$$

Assume now that in addition $\mathrm{Tr}(\widehat{\rho}) = 1$; if $\widehat{\rho}$ is positive then it is called a *density matrix*. If A and B are two essentially self-adjoint operators defined on some common dense subset of $L^2(\mathbb{R}^N)$ the *covariance* of the pair (A, B) with respect to $\widehat{\rho}$ is by definition

$$\Delta(A,B)_{\widehat{\rho}} = \tfrac{1}{2}\left\langle AB + BA\right\rangle_{\widehat{\rho}} - \left\langle A\right\rangle_{\widehat{\rho}}\left\langle B\right\rangle_{\widehat{\rho}} \tag{4}$$

where $\left\langle A\right\rangle_{\widehat{\rho}} = \mathrm{Tr}(A\widehat{\rho})$, and so on. Choosing in particular for A the position operator $X_j = x_j$ and for B the momentum operator $P_j = -i\hbar\partial/\partial x_j$ the *covariance matrix* is the symmetric $2N \times 2N$ matrix

$$\Sigma_{\widehat{\rho}} = \begin{pmatrix} \Delta(X,X)_{\widehat{\rho}} & \Delta(X,P)_{\widehat{\rho}} \\ \Delta(P,X)_{\widehat{\rho}} & \Delta(P,P)_{\widehat{\rho}} \end{pmatrix} \tag{5}$$

where $\Delta(X,X)_{\widehat{\rho}} = (\Delta(X_j,X_k)_{\widehat{\rho}})_{1\leq j,k\leq N}$, $\Delta(X,P)_{\widehat{\rho}} = (\Delta(X_j,P_k)_{\widehat{\rho}})_{1\leq j,k\leq N}$ and so on (we assume that all second moments exist; this condition is satisfied for instance if $(1+|z|^2)\rho$ is absolutely integrable). The covariance matrix is a fundamental object in both classical and quantum statistical mechanics because it incorporates the correlations between the considered variables. The Robertson–Schrödinger uncertainty principle says that

$$(\Delta X_j)^2_{\widehat{\rho}}(\Delta P_j)^2_{\widehat{\rho}} \geq \Delta(X_j,P_j)^2_{\widehat{\rho}} + \tfrac{1}{4}\hbar^2 \ ,\ 1 \leq j \leq N \tag{6}$$

$$(\Delta X_j)^2_{\widehat{\rho}}(\Delta P_k)^2_{\widehat{\rho}} \geq \Delta(X_j,P_k)^2_{\widehat{\rho}} \ \text{ for } \ j \neq k. \tag{7}$$

where $\Delta(X,P)$ is the covariance of the pair (X_j, P_j) (see the original articles [10, 11] and the historical discussion by Trifonov and Donev [12]; for a "modern" proof in the general case of non-commuting observables the reader could consult Messiah's classical treatise [13]).

One has the following fundamental result well-known in quantum optics, and used in the study of entanglement and separability (see for instance [14, 15]):

> (I) *If the self-adjoint trace-class operator* $\widehat{\rho}$ with $Tr(\widehat{\rho}) = 1$ *is positive, that is if it is a density matrix, then the Hermitian matrix* $\Sigma_{\widehat{\rho}} + \frac{i\hbar}{2}J$ *is positive semi-definite*:
>
> $$\Sigma_{\widehat{\rho}} + \frac{i\hbar}{2}J \geq 0. \tag{8}$$

(That $\Sigma_{\widehat{\rho}} + \frac{i\hbar}{2}J$ is Hermitian results from the symmetry of $\Sigma_{\widehat{\rho}}$ and the fact that $J^T = -J$). The relation between property (I) and the Robertson–Schrödinger uncertainty principle is the following:

> (II) *Condition (8) is equivalent to the Robertson–Schrödinger inequalities (6)–(7).*

That the formulation (8) of the uncertainty principle is invariant under linear canonical transformations follows at once from the fact that S is a symplectic matrix; this makes the superiority of this formulation on the usual one: it replaces the quite complicated and tedious verification of the inequalities (6)–(7) by the calculation of a set of eigenvalues. To see why, let us begin by giving two definitions. Let M be any real positive-definite $2N \times 2N$ matrix. Since JM is equivalent to the antisymmetric matrix $M^{1/2}JM^{1/2}$ its eigenvalues are of the type $\pm i\mu_j$ $(j = 1, ..., N)$ with $\mu_j > 0$. Ordering the μ_j so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ we call the sequence $(\mu_1, ..., \mu_N)$ the *symplectic spectrum* of M; the number $\mu = \mu_1$ is called the *Williamson invariant* of M. Williamson [17] has proved that that there exists a $2N \times 2N$ symplectic matrix such that $M = S^T DS$ with

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \quad , \quad \Lambda = \mathrm{diag}(\mu_1, ..., \mu_N) \tag{9}$$

("Williamson diagonal form"). Now, one proves [14, 15] (see [8] for a detailed exposition), that condition (8) (and hence the Robertson–Schrödinger inequalities) is equivalent to:

> (III) *The Williamson invariant μ of the covariance matrix $\Sigma_{\widehat{\rho}}$ satisfies $\mu \geq \frac{1}{2}\hbar$.*

We emphasize that the equivalent conditions (I)–(III) are not sufficient to ensure positivity; an illustration is the example of Narcowich and O'Connell [4] mentioned in the Introduction. It goes as follows: let the function $W(x,p)$ be determined by its Fourier transform via

$$\int e^{i(xx'+pp')} W(x',p')dp'dx' = (1 - \tfrac{1}{2}\alpha x^2 - \tfrac{1}{2}\beta p^2)e^{-(\alpha^2 x^4 + \beta^2 p^4)} \tag{10}$$

(with $\alpha, \beta > 0$). It is easily checked that W is real and that the corresponding operator $\widehat{\rho}$ satisfies $\mathrm{Tr}(\widehat{\rho}) = 1$. Narcowich and O'Connell then show that the uncertainty principle is satisfied as soon as α and β are chosen such that $\alpha\beta \geq \hbar^2/4$. However, even with that choice, the operator $\widehat{\rho}$ is never non-negative because the average of p^4 is in all cases given by

$$\int p^4 W(x,p)dxdp = -24\alpha^2 < 0; \tag{11}$$

ρ can thus not be the density matrix of any quantum state.

The considerations above explain the difficulties with positivity questions occurring when one rescales Wigner distributions as Man'ko *et al* do in [3]. Let in fact z_α denote any of the components of the vector $z = (x_1, ..., x_N; p_1, ..., p_N)$ and let Z_α, Z_β be the operators corresponding to z_α, z_β. Defining the rescaled Wigner distribution W^λ by

$$W^\lambda(z) = \lambda^{-2N} W(\lambda z) \quad , \quad \lambda > 0 \tag{12}$$

we have

$$\int W^\lambda(z)d^{2N}z = \int W(z)d^{2N}z = 1 \tag{13}$$

and a straightforward calculation shows that

$$\Delta(Z_\alpha, Z_\beta)_{\widehat{\rho}^\lambda} = \frac{1}{\lambda^2}\Delta(Z_\alpha, Z_\beta)_{\widehat{\rho}} \tag{14}$$

where $\widehat{\rho}^\lambda$ is the operator corresponding to W^λ, hence

$$\Sigma_{\widehat{\rho}^\lambda} = \frac{1}{\lambda^2}\Sigma_{\widehat{\rho}}. \tag{15}$$

Let μ and μ^λ be the Williamson invariants of the matrices $\Sigma_{\widehat{\rho}}$ and $\Sigma_{\widehat{\rho}^\lambda}$, respectively. In view of condition (III) in last section, we must have $\mu \geq \frac{1}{2}\hbar$. If now $\widehat{\rho}^\lambda$ is also to be a density operator we must have $\mu^\lambda \geq \frac{1}{2}\hbar$ as well, that is, equivalently $\mu \geq \frac{1}{2}\lambda^{-2}\hbar$. This requires that $\lambda \leq 1$.

In the Introduction section of this Letter we referred to necessary and sufficient conditions for a self-adjoint trace-class operator to be positive. In fact, Kastler [18] and Loupias and Miracle-Sole [19, 20] have shown that the operator $\widehat{\rho}$ is positive (and hence a density operator) if and only the following so-called "KLM conditions" hold: for every integer $m = 1, 2, ...$ the complex matrix $F = (F_{jk}(z_j, z_k))_{1\leq j,k\leq m}$ with

$$F_{jk}(z_j, z_k) = e^{\frac{i\hbar}{2}\sigma(z_j,z_k)}\mathcal{F}_\sigma W(z_j - z_k) \tag{16}$$

is positive semi-definite; here

$$\mathcal{F}_\sigma W_\rho(z) = \int e^{i\sigma(z,z')}W(z')d^N z' \tag{17}$$

is the symplectic Fourier transform of the Wigner distribution ρ.

In [16], Lemma 2.1, Narcowich shows that the KLM conditions imply that $\Sigma_{\widehat{\rho}} + \frac{i\hbar}{2}J \geq 0$. In view of (I), (II) above we thus have the following *necessary* condition for $\widehat{\rho}$ to be positive:

> (IV) *Assume that the operator $\widehat{\rho}$ is positive; then its covariance matrix must satisfy condition (8) or, equivalently, the Schrödinger–Robertson inequalities (6)–(7).*

This statement thus fully justifies and completes the statement of Man'ko et al. [3] that "*...the uncertainty principle does not determine the quantum state*".

3 Gaussian estimates and Hardy's theorem

We ask the following question:

> "*Under which conditions on M can a function W such that $W(z) \leq Ce^{-\frac{1}{\hbar}Mz\cdot z}$ be the Wigner distribution of some mixed sate?*".

The answer to that question is given by following theorem, the proof of which relies on the following old result due to Hardy [5]: assume that the square-integrable function ψ and its Fourier transform

$$F\psi(p) = \left(\tfrac{1}{2\pi\hbar}\right)^N \int e^{-\frac{i}{\hbar}p\cdot x}\psi(x)d^N x$$

are such that $|\psi(x)| \leq Ce^{-\frac{a}{2\hbar}|x|^2}$ and $|F\psi(p)| \leq Ce^{-\frac{b}{2\hbar}|p|^2}$ $(a,\ b > 0)$. Then we must have $ab \leq 1$, and if $ab = 1$ then $\psi(x) = Ae^{-\frac{a}{2\hbar}|x|^2}$ for some complex constant A. If $ab > 1$ then $\psi = 0$.

Theorem 1 *Let $\widehat{\rho}$ be a density operator and assume that its Wigner distribution W satisfies an estimate*

$$W(z) \leq Ce^{-\frac{1}{\hbar}Mz\cdot z} \tag{18}$$

where $C > 0$ is some constant and $M = M^T > 0$. Then the Williamson invariant μ of M must satisfy $\mu_1 \leq 1$. Equivalently: the matrix $\Sigma_{\widehat{\rho}} = \frac{\hbar}{2}M^{-1}$ must satisfy the uncertainty principle: $\Sigma_{\widehat{\rho}} + \frac{i\hbar}{2}J \geq 0$.

To prove this we will use the fact that there exists an orthonormal system of vectors $(\psi_j)_j$ in $L^2(\mathbb{R}^N)$ such that

$$W(z) = \sum_j \alpha_j W\psi_j(z) \tag{19}$$

with $\sum_j \alpha_j = 1$, $\alpha_j > 0$ (see e.g. [8] and the references therein). Let S be a symplectic matrix such that $M = S^T DS$ with D as in (9); then the inequality (18) is equivalent to

$$W_\rho(S^{-1}z) \leq Ce^{-\frac{1}{\hbar}\Lambda x\cdot x}e^{-\frac{1}{\hbar}\Lambda p\cdot p}. \tag{20}$$

Integrating successively with respect to the variables p_j and x_j we get

$$\int W(S^{-1}z)dp \leq C_\Lambda e^{-\frac{1}{\hbar}\Lambda x\cdot x} \quad , \quad \int W(S^{-1}z)d^N x \leq C_\Lambda e^{-\frac{1}{\hbar}\Lambda p\cdot p} \tag{21}$$

with $C_\Lambda = C\int e^{-\frac{1}{\hbar}\Lambda x\cdot x}d^N x$. We next observe that

$$W(S^{-1}z) = \sum_j \alpha_j W\psi_j(S^{-1}z) = \sum_j \alpha_j W(\widehat{S}\psi_j)(z) \tag{22}$$

where $\widehat{S}$ is any of the two metaplectic operators corresponding to S (see for instance [8, 9] and the references therein). Taking into account the formulae

$$\int W(\widehat{S}\psi_j)(z)d^N p = |\widehat{S}\psi_j(x)|^2 \ , \ \int W(\widehat{S}\psi_j)(z)d^N x = |F(\widehat{S}\psi_j)(p)|^2 \tag{23}$$

it follows that we have

$$\sum_j \alpha_j|\widehat{S}\psi_j(x)|^2 \leq C_\Lambda e^{-\frac{1}{\hbar}\Lambda x\cdot x} \quad , \quad \sum_j \alpha_j|F(\widehat{S}\psi_j)(p)|^2 \leq C_\Lambda e^{-\frac{1}{\hbar}\Lambda p\cdot p} \tag{24}$$

and hence, in particular,

$$|\widehat{S}\psi_j(x)| \leq C_{j,\Lambda} e^{-\frac{1}{2\hbar}\Lambda x \cdot x} \quad , \quad |F(\widehat{S}\psi_j)(p)| \leq C_{j,\Lambda} e^{-\frac{1}{2\hbar}\Lambda p \cdot p} \tag{25}$$

with $C_{j,\Lambda} = \sqrt{C_\Lambda / \alpha_j}$. Since $\Lambda = \mathrm{diag}(\mu_1, ..., \mu_N)$ with $\mu_1 \geq \cdots \geq \mu_N$ it follows that

$$|\widehat{S}\psi_j(x)| \leq C_{j,\Lambda} e^{-\frac{1}{2\hbar}\Lambda x \cdot x} \quad , \quad |F(\widehat{S}\psi_j)(p)| \leq C_{j,\Lambda} e^{-\frac{1}{2\hbar}\Lambda p \cdot p}. \tag{26}$$

We claim that these inequalities can only hold if $\mu_1 \leq 1$. Set

$$\phi_j(x_1) = \widehat{S}\psi_j(x_1, 0, ..., 0).$$

By the first inequality (26) we have

$$|\phi_j(x_1)| \leq C_{j,\Lambda} e^{-\frac{\mu_1}{2\hbar}x_1^2}. \tag{27}$$

Denoting by F_1 the Fourier transform in the variable x_1 a straightforward calculation shows that

$$\int F(\widehat{S}\psi_j)(p) dp_2 \cdots dp_N = (2\pi\hbar)^{(N-1)/2} F_1\phi_j(p_1). \tag{28}$$

and hence, in view of the second inequality (26),

$$|F_1\phi_j(p_1)| \leq \left(\tfrac{1}{2\pi\hbar}\right)^{(N-1)/2} C_{j,\Lambda} \int e^{-\frac{1}{2\hbar}\sum_{j=1}^N \mu_j p_j^2} dp_2 \cdots dp_N \tag{29}$$

that is

$$|F_1\phi_j(p_1)(p_1)| \leq C_{j,\Lambda} e^{-\frac{\mu_1}{2\hbar}p_1^2} \tag{30}$$

for a new constant $C_{j,\Lambda}$. Applying Hardy's theorem to (27) and (30) we must have $\mu_1^2 \leq 1$, which proves our claim.

The result above shows the reason for which the rescaling can be used to produce negative operators from a positive one: assume that $W_\rho(z) \leq Ce^{-\frac{1}{\hbar}Mz \cdot z}$; then

$$\lambda^{-2N} W_\rho(\lambda z) \leq \lambda^{-2N} C_\lambda e^{-\frac{1}{\hbar}M_\lambda z \cdot z} \tag{31}$$

with $C_\lambda = \lambda^{-2N}C$ and $M_\lambda = \lambda^2 M$. The Williamson invariant of M_λ is $\lambda^2\mu_1$ and the condition $\lambda^2\mu_1 \leq 1$ will be violated as soon as we choose $\lambda > 1/\sqrt{\mu_1}$. (In [3] Man'ko *et al.* work in units in which $\hbar = 1$; it is therefore not immediately obvious that the procedure they implement to construct non-positive operators by rescaling coordinates in a non-symplectic way is tantamount to increasing the value of Planck's constant so that the uncertainty principle is violated.) In fact, one immediately understands why the rescaling procedure of Man'ko *et al.* works: it consists (in the example they consider) in replacing M by a matrix that is "too small". In addition, as a by-product of our result we recover the property that the support of a Wigner distribution can never be bounded in phase space.

Remark 2 *Theorem 1 also allows us to recover in a simple way a result of Folland and Sitaram [25]: a Wigner distribution can never be compactly supported. Suppose indeed that there exists some $R > 0$ such that $W_\rho(z) = 0$ for $|z| \geq R$. For any given μ we can always choose $C > 0$ large enough so that $W_\rho(z) \leq Ce^{-\frac{\mu}{\hbar}|z|^2}$ for all z. Choosing $\mu > 1$ this contradicts the theorem..*

4 Relation with Quantum Blobs

In recent previous work [6, 7] one of us has introduced the notion of "quantum blob" and of "admissible ellipsoid" in connection with the study of a coordinate-free formulation of the uncertainty principle. A quantum blob is the image of a phase-space ball with radius $\sqrt{\hbar}$ by a (linear or affine) symplectic translation. An admissible ellipsoid is a phase-space ellipsoid containing a quantum blob. Characteristic properties are:

- The section of a quantum blob by any plane through its center which is parallel to a plane of conjugate coordinates x_j, p_j has area $\frac{1}{2}h$;
- A phase-space ellipsoid is admissible if and only if its section by any plane through its center which is parallel to a plane of conjugate coordinates x_j, p_j has area at least $\frac{1}{2}h$.

Moreover:

> (V) *An ellipsoid* $\mathcal{B}_M : Mz \cdot z \leq \hbar$ is admissible if and only if $c(\mathcal{B}_M) \geq \frac{1}{2}h$, c any symplectic capacity [21] on $\mathbb{R}^{2N}$, and this condition is equivalent to $\Sigma + \frac{i\hbar}{2}J \geq 0$ with $\Sigma = \frac{\hbar}{2}M^{-1}$.

and

> (VI) *The symplectic capacity of* $\mathcal{B}_M : Mz \cdot z \leq \hbar$ is $c(\mathcal{B}_M) = \pi\hbar/\mu_1$ where $(\mu_1, ..., \mu_N)$ is the symplectic spectrum of $\mathcal{B}_M$ (see [8, 21]).

We can thus re-express Theorem 1 in the following coordinate-free form:

Theorem 3 *Assume that the Wigner distribution of a density operator* $\widehat{\rho}$ *is such that* $W(z) \leq Ce^{-\frac{1}{\hbar}Mz\cdot z}$. *Then* $c(\mathcal{B}_M) \geq \frac{1}{2}h$ *where* $\mathcal{B}_M$ *is the ellipsoid* $Mz \cdot z \leq \hbar$

It can be interpreted in a very visual way as follows: assume that we have coarse-grained phase space by quantum blobs $S(B(\sqrt{\hbar}))$. Then the Wigner ellipsoid of a density operator cannot be arbitrarily small, but must contain such a quantum blob. Equivalently: the Wigner ellipsoid must be defined on the "quantum phase-space" consisting of all parts of $\mathbb{R}^{2N}$ containing a quantum blob.

5 Conclusion and Comments

A "simple" characterization of positivity for trace-class operators is still to be found. We have given one such characterization for a particular class of putative Wigner distributions (those dominated by a phase-space Gaussian). In the general case possibly the phase-space techniques and concepts (symplectic

capacities) developed in [6, 7] could provide further insight about what such a condition could be (cf. Theorem 3 above). The methods proposed in Bohm and Hiley [22] could perhaps shed some light on the question; also see Bracken and Wood [23] who introduce the interesting notion of "Groenewold operator" to study positivity (but from a slightly different point of view).

We finally remark that in the discussed paper [3] Man'ko *et al.* use the notion of quantum fidelity (which, besides, plays an important role in the study of Loschmidt echo) to prove that the Wigner function of the first excited state of the oscillator does not lead to a positive operator when rescaled; perhaps their idea could be exploited in a more general context to shed some light on the difficult question of positivity? We will come back to these fundamental questions in a forthcoming paper.

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